Gridding of geophysical field data using finite element methods
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Summary
The most commonly used interpolation technique in geophysics is minimum curvature gridding. The majority of current algorithms and commercial packages invariably employ the finite difference to obtain the solution. This approach has been used extensively for over 25 years because of its simplicity and availability. However, finite difference method is not suited for handling irregularly spaced data because the uniformly partitioned mesh in finite difference solution is fundamentally incompatible with the random nature of the data locations. As a result, the finite difference solution of minimum curvature gridding can suffer from slow convergence and poor gridding accuracy. To address this difficulty, we have developed a new approach using a finite element method. By employing a triangular mesh adapted to irregularly located observation points, the new approach ensures the accuracy of the gridded data. In addition, we use a mixed finite element formulation to solve the problem efficiently. The new method therefore overcomes the two major drawbacks present in the finite difference approach. In this presentation, we will outline the theoretical formulation and numerical solution of this approach and illustrate it with both synthetic and field data sets.

Introduction
A crucial step in geo-potential data processing is gridding, which interpolates observations located irregularly in space onto a uniform grid. This step is the basis for subsequent processing and displaying such as Fourier domain filtering, contouring, and shaded-relief imaging. However, producing reliable gridded data, one needs to design numerical methods which are accurate, fast and stable. Since its introduction (Briggs, 1974), minimum curvature gridding has been the method of choice for gridding geo-potential data. This method constructs an interpolation surface \( u(x, y) \) that minimizes the total curvature,

\[
C(u) = \int_0^1 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)^2 \, dx \, dy,
\]

subject to the condition that the surface passes through the observed data \( u(x_i, y_i) = d_i, \quad i = 1, \cdots, n \). With appropriate boundary conditions, this problem is equivalent to solving the bi-harmonic equation

\[
\nabla^4 u = 0
\]

with the data constraints as the interior boundary condition, where \( \nabla^2 = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \).

Because of historical reasons, and the simplicity of the method, the finite difference solution of minimum curvature gridding has been used for more than a quarter of a century. To our knowledge, all current freeware and commercial packages use the finite difference scheme proposed originally by Briggs (1974) to solve the bi-harmonic equation. This approach has two major drawbacks that have long been recognized by practicing geophysicists. First, the finite difference method uses a mesh consisting of rectangular elements and the nodal points generally do not coincide with the observations to be gridded. An approximation based on a Taylor series expansion or simple shifting is often used to link the data value at the nodal points to those at the observation locations. This often leads to large errors in the gridded data. Second, the solution is obtained by successive approximation and the convergence is often slow. In practical use, the algorithm is often terminated by a limited number of iterations. As a result, the final solution is not necessarily the optimal one sought through the theoretical formulation.

Many advances have been made since the early days of the finite difference approach. One of them is Galerkin finite element method (FEM). This is a general technique that offers flexibility, efficiency, and stability for numerical solutions of differential and integral equations. It has been applied to many geophysical problems to obtain numerical solutions that are otherwise difficult to solve. It seems natural that this method should lend itself to a modern solution of the age-old minimum curvature gridding problem.

In the research reported in this presentation, we have reformulated the minimum curvature gridding problem using a finite element scheme and developed a new algorithm that takes advantage of this method. In particular, the FEM solution can fit the observed data exactly or pass through the observations within a user-prescribed error tolerance. Furthermore, FEM enables the subdivision of a large data set into smaller subsets based on the finite element partition. Consequently, gridding of large data sets can be solved efficiently through domain decomposition.

In our formulation, the observation points always coincide with nodal points of the finite element mesh. This ensures a high degree of accuracy in fitting the observed data. In addition, we solve the problem using a mixed finite element method that produces an interpolation surface with continuous second order derivatives. Numerically, this formulation results in a symmetric positive definite system, which can be solved efficiently using the conjugate gradient method.
Methodology

Minimum curvature gridding seeks to construct a surface $u = u(x, y)$ on a region $\Omega$ in $\mathbb{R}^2$ that minimizes the following total squared-curvature in eq.(1):

$$C(u) = \int_{\Omega} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)^2 dxdy,$$

subject to the condition $u(x_i, y_i) = d_i, \quad i = 1, \ldots, n,$ where $(x_i, y_i)$ are given observation locations with data values $d_i$ in the region $\Omega$.

In order to derive a differential equation for the above minimization problem, we introduce a few functional spaces necessary for the analysis. Let $L^2(\Omega)$ be the space of square-integrable functions given by

$$L^2(\Omega) = \{ v : \Omega \to \mathbb{R}, \quad \int_{\Omega} v^2 dxdy < +\infty \},$$

and $G$ be the functional space

$$G = \{ v : \partial^j v \in L^2(\Omega), \quad j = 0, 1, 2, \quad \frac{\partial v}{\partial n} = 0 \text{ on } \partial \Omega \},$$

where $\partial^j$ denotes the $j$-th order partial derivatives and $\frac{\partial}{\partial n}$ represents a derivative along the outward normal vector of the boundary $\partial \Omega$ of region $\Omega$.

Our method is based on minimizing the total-squared-curvature in $G$. More precisely, the minimization problem seeks $u \in G$ such that $u(x_i, y_i) = d_i$ and

$$C(u) \leq C(v), \quad \text{for all } v \in G, \text{ and } v(x_i, y_i) = d_i, \quad i = 1, \ldots, n.$$

Using calculus of variation, we can show that the solution $u \in G$ to the minimization problem in eq.(3) satisfies $u(x_i, y_i) = d_i$ and

$$\int_{\Omega} \nabla^2 u \nabla^2 v dxdy + \sum_{i=1}^{n} \lambda_i v(x_i, y_i) = 0, \quad \forall v \in G,$$  \quad \text{(4)}

where $\lambda_i (i = 1, 2, \ldots, n)$ are known as Lagrange multipliers. The variational problem in eq.(4) has a unique solution. To this end, it suffices to show that the homogeneous case (i.e., $d_i = 0$) has only trivial solution $u \equiv 0, \lambda_i = 0$. Since $d_i = 0$, setting $v = u$ in eq.(4) yields $\nabla^2 u = 0$. This, together with $\frac{\partial u}{\partial n} = 0$ on $\partial \Omega$ and $u(x_i, y_i) = 0$ implies $u$ can only be the trivial solution. With $u \equiv 0$, eq. (4) is reduced to

$$\sum_{i=1}^{n} \lambda_i v(x_i, y_i) = 0, \quad \forall v \in G,$$

which implies $\lambda_i = 0$ by choosing a particular test function $v \in G$. As a result, eq.(4) must have a unique solution to the minimization problem in eq.(3) must be the solution to eq.(4). Therefore, the minimization in eq.(3) has a unique solution.

A mixed finite element method is employed to approximate the solution to eq.(4) in the following four steps:

**Step 1**, derive a mixed variational formulation for eq.(4) by introducing a new variable $w = \nabla^2 u$. Substituting $w$ into eq.(4), we obtain

$$\int_{\Omega} \nabla w \nabla v dxdy - \sum_{i=1}^{n} \lambda_i v(x_i, y_i) = 0, \quad v \in G,$$  \quad \text{(5)}

where we have used integration by parts in the first term of eq.(5). Let $H^1(\Omega)$ be the Sobolev space consisting of functions whose derivatives up to order 1 are square integrable. Next, we test $w = \nabla^2 u$ against any function $\varphi \in H^1(\Omega)$ to obtain

$$\int_{\Omega} w \varphi dxdy + \int_{\Omega} \nabla u \nabla \varphi dxdy = 0, \quad \varphi \in H^1(\Omega).$$  \quad \text{(6)}

Eqs (5) and (6) comprise the mixed variational formulation for eq.(4).

**Step 2**, construct a finite element partition $T_h$, such that the observation points are nodal points of $T_h$. This can be accomplished by connecting the observation points into triangles according to the Delaunay criterion (Shewchuk, 1996), and then refine the triangulation by inserting additional nodal points (Jim, 1995). The Delaunay triangulation ensures that neither the area nor the angles of any triangle are too large or too small. A typical example of the finite element partition $T_h$ is shown in Figure 1.

**Step 3**, construct a finite element space $S_h$ consisting of continuous piecewise polynomials of degree $r \geq 1$ on $T_h$, and obtain numerical solutions $u_h \in S_h$ and $w_h \in S_h$ by solving

$$(\nabla u_h, \nabla \varphi) + (u_h, \varphi) = 0, \quad \forall \varphi \in S_h,$$

$$(\nabla w_h, \nabla \psi) = 0, \quad \forall \psi \in S_h, \quad \psi(x_i, y_i) = 0$$  \quad \text{(7)}

with the constraint $u(x_i, y_i) = d_i$ for all $i = 1, 2, \ldots, n$. Here, $(\cdot, \cdot)$ denotes the standard $L^2$ inner product in $L^2(\Omega)$.
Finite Element Gridding

Step 4, formulate a matrix problem for eq.(7) and solve it using a preconditioned conjugate gradient method. For piecewise linear finite elements, the matrix problem is formulated by using the standard nodal basis \( \{ \phi_1, \phi_2, \ldots, \phi_m \} \) (Mitchell, 1977), where \( \phi_i \) is defined as

\[
\phi_i(p_j) = \begin{cases} 
1, & \text{if } i = j \\
0, & \text{if } i \neq j 
\end{cases}
\]

for \( i = 1, 2, \ldots, m \). Here, \( m \) is the total number of nodal points and \( p_j \) is the coordinates for the \( j \)th nodal point of the finite element partition \( T_h \). It is well-known that

\[
u_h(x, y) = \sum_{i=1}^{m} \xi_i \phi_i(x, y), \quad \xi_i = u_h(x_i, y_i),
\]

and

\[u_h(x, y) = \sum_{i=1}^{m} \zeta_i \phi_i(x, y), \quad \zeta_i = w_h(x_i, y_i).\]

Substituting eqs (8) and (9) into eq. (7) and taking \( \varphi = \phi_i \) and \( v = \phi_i \) \( (i = 1, 2, \ldots, m) \), respectively, we derive the linear system of equations for the coefficients \( \{ \xi_i \}_{i=1}^{m} \) and \( \{ \zeta_i \}_{i=1}^{m} \) from the variational form of the mixed finite element in eq.(7). This system of equations is symmetric. Finally, we eliminate the Lagrange multiplier \( \lambda_i \) and the new variable \( w_h \) and solve the resulting matrix-vector problem using the conjugate gradient method.

After obtaining the coefficients \( \{ \xi_i \}_{i=1}^{m} \) by solving the matrix-vector problem, we have a numerical solution \( u_h(x, y) = \sum_{i=1}^{m} \xi_i \phi_i(x, y) \). This is a complete surface and provides numerical values for \( u(x, y) \) at every point \( (x, y) \) in the region \( \Omega \). Furthermore, the numerical solution \( w_h \) provides a continuous and very accurate approximation for the second order derivative \( w = \nabla^2 u \).

Examples

We now illustrate the newly developed method by numerical examples. The first is a simulated data set with randomly distributed magnetic data. The second data set simulates a line-based magnetic survey. For brevity, we will not reproduce tests on field data here, but will present them at the meeting.

Both examples are generated from a synthetic magnetic model. The model is composed of susceptible blocks designed to simulate anomalies of different shapes and magnitudes. Using synthetic data also allows us to compare gridded result with the true data.

The test results for the random data set are shown in Figure 2. We have simulated 1000 randomly located observations from the synthetic model within an area of 2000 m by 2000 m. The locations are shown as pluses (+) in Figure 2(a). We then use the new FEM method to interpolate these data to a 20 m by 20 m grid. The gridded

![Fig. 2: Example of randomly located data. Panel (a) shows the data points as pluses (+) and the gridding results in contour lines. The contour interval is 5 nT. Panel (b) shows the percentage difference between the gridded and true values.](image1)

![Fig. 3: Example of profile data. Panel (a) shows the data points as pluses (+) and the gridding results in contour lines. The contour interval is 5 nT. Panel (b) shows the percentage difference between the gridded and true values. The gridding results and error are very similar to those from the random data example in Figure 2.](image2)
data are displayed as contours in Figure 2(a). The contour lines are smooth and there are no erratic variations. To examine the error distribution, we display the percentage difference map between gridded and true data in Figure 2(b). The majority of the difference map is close to zero, while large errors occur in areas of high gradients. This is to be expected since the minimum curvature requirement is not necessarily consistent with the frequency content of the data. The overall relative error of the difference, defined by the norm of the difference divided by the norm of the true data, is 0.6%.

Figure 3 shows the test result for simulated profile data. The nominal line spacing is 100 m and there is a variation of 30 m. The station spacing ranges from 16 to 24 m. There is a total of 2331 data points. We then interpolate these data to the same 20 m by 20 m grid as in the first example. The original data locations and gridded data are shown in Figure 3(a) as plus (+) and contour lines respectively. The error map with the true data is shown in Figure 3(b). We can make two observations. First, the error map is consistent with that in Figure 2(b), showing relatively large errors again in the areas of high gradients. Second, the gridded data are very similar to that from the random data locations. These observations demonstrate the stability and robustness of the algorithm when working with different types of data locations. The overall relative difference with the true data is 0.5% in this test.

Conclusions

We have developed a mixed finite element method for solving the minimum curvature gridding problem. This method is well suited for gridding arbitrarily located observations in irregularly shaped areas. Furthermore, the method generates an interpolation surface that has continuous second derivatives and passes over the observations exactly if required. Numerical tests have demonstrated the stability and efficiency of the algorithm. We have also obtained an error bound for a general finite element space, and shown that, with a good domain partition, this method has at least a linear convergence. Therefore, our new approach based on a finite element method offers a viable and superior alternative to the traditional minimum curvature gridding based on a finite difference method.

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